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INSTABILITY OF PLANING SURFACES

T. Francis Ogilvie

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THE DEPARTMENT OF NAVAL ARCHITECTURE AND MARINE ENGINEERING

THE UNIVERSITY OF MICHIGAN
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ABSTRACT

A mathematical analysis is developed for the hydrodynamic problem of a two-dimensional planing surface which is heaving sinusoidally. From the assumptions of small angle of attack and small amplitudes of motion, it is possible to formulate a linear problem. Gravity is neglected. A condition is developed for predicting instability of the steady forward motion.

INTRODUCTION

Several years ago, Mottard (1965) reported some observations of planing instability involving just one mechanical degree of freedom. He towed a large-aspect-ratio planing surface in such a way that it could not pitch, although it was free to heave. In a series of careful experiments, he found that there was a rather clearly defined range of conditions under which the planing surface oscillated spontaneously. The instability appeared to be quite similar to the flutter of an airfoil.

The occurrence of flutter normally requires that two vibrational degrees of freedom be involved, for otherwise the hydrodynamic force provides positive damping, and a spontaneous oscillation cannot develop. Mottard checked carefully in his experiments to determine that no pitching motion occurred or, at least, that pitching motion had no effect on the phenomenon he was observing. His planing surface underwent a spontaneous oscillation in heave alone.

Mottard suggested that there was effectively a second degree of freedom because of the presence of the free surface. Ahead of the planing surface, the free surface oscillates at the same frequency as the planing surface itself but generally with a phase shift. Thus the location of the leading edge of the planing surface varies for two reasons: (1) the planing-surface immersion varie in time, and (2) the free-surface elevation just ahead of the planing surface varies. Mottard showed that his hypothesis was consistent with his measured values of lift coefficient, lift-curve slope, etc.

In the present paper, a first attempt is made to analyze the hydrodynamics of this problem. It is assumed that a two-dimensional flat planing surface is moving at constant forward speed and oscillating sinusoidally in heave. Gravity is neglected with respect to both the steady and unsteady components of motion; this leads to some difficulty, as might be expected. The location of the leading edge is an unknown quantity in the problem.

It is assumed that the lifting-surface problem can be linearized in the manner of Wagner (1932). Then the free-surface problem reduces to an airfoil problem which is solved in terms of the unknown variable location of the leading edge. The free-surface disturbance ahead of the planing surface is determined, also in terms of the unknown location of the leading edge. Finally, this unknown quantity is determined by a simple geometrical matching of the water elevation to the instantaneous position of the planing surface. At this point, the linear hydrodynamic problem is completely solved, and one can compute the lift, which is then resolved into components in phase with displacement and with velocity. If the latter is positive, a free oscillation will tend to grow until nonlinear phenomena change the problem in some way.

In terms of an equivalent airfoil problem, we may say that the two degrees of freedom are the heave motion and the variation in chord length which follows from the movement of the leading edge. In planing problems, of course, the leading edge is likely to be rather poorly defined, since there is an extended region filled with spray. In an idealized treatment of the planing problem, there is a jet of water thrown forward, rather than spray, but even this idealization does not help in defining the location of the leading edge. However, in the linear theory the location of the leading edge is uniquely defined, for the jet thickness is a small quantity of second order in terms of, say, angle of attack. Similarly, the entire region in which a portion of the incident flow is reversed to form the jet is a very small region, its characteristic dimensions being second-order quantities. As Wagner showed, the flow in this region appears from a distance to have a square-root singularity entirely comparable to the square-root singularity at the leading edge of an idealized airfoil. These results can also be derived in a more rigorous manner from the nonlinear analysis of Green (1935).

It has long been recognized that the analytical treatment of planing surfaces on water of infinite depth leads to an anomalous result if gravitation effects are neglected: The height of the free surface drops off logarithmically at infinity both upstream and downstream. This anomaly has been studied thoroughly by Rispin (1966) and Wu (1967). They showed that the usual analysis is really valid only in a near-field sense; it must be matched to a far-field description which includes the presence of gravity waves. Thus the classical gravity-free description of planing is not incorrect, but it must be interpreted carefully.

Gravity is neglected in the present paper also. Since our solution includes the steady-motion problem as a special case, we must anticipate a similar difficulty. In fact, the anomalous behavior at infinity becomes critical when we seek to find the free-surface disturbance ahead of the planing surface. This will be discussed further in its proper place.

We do not attempt to develop inner and outer solutions of the time-dependent problem, and so it is not surprising that the solution is poorly behaved near zero frequency. In fact, our solution can really be interpreted only in terms of generalized functions, because of the difficulties discussed, and this is a rather unsatisfactory state of affairs for a solution which is supposed to have physical meaning. Nevertheless, there is reason to believe that the solution does have meaning for frequencies not too near zero.

THE BOUNDARY-VALUE PROBLEM

We assume that there is a steady flow incident from the left, with speed U. The planing surface is flat, with an angle of attack α in steady motion. The trailing edge is located at the point $(x,y) = (1,d)$ in steady motion, the leading edge at $(0, \alpha + d)$. In the general problem, we assume that the planing surface is displaced vertically a distance $h = h(t)$, and so the equation of the planing surface is:

$$y = (1 - x)\alpha + d + h(t) \quad . \tag{1}$$

The planing surface has a sharp after termination at $x = 1$, but we do not specify a forward termination; the flat surface extends indefinitely far forward.

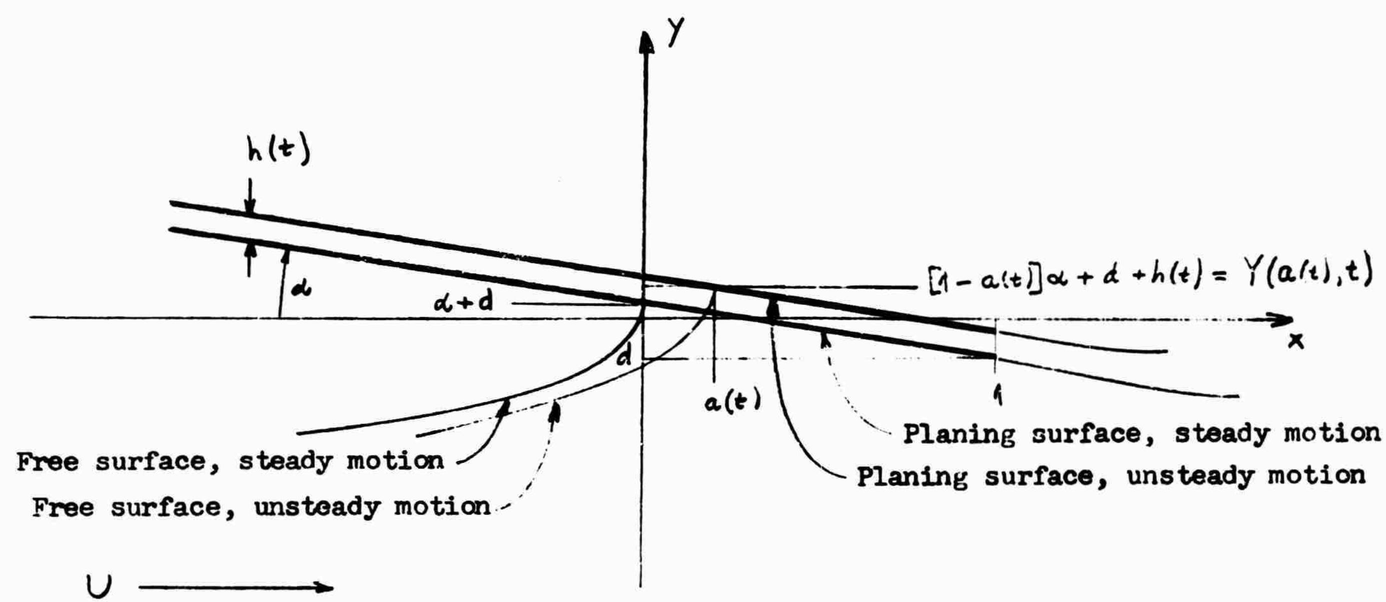
The form of the free surface is described for the moment only by an unknown function:

$$y = Y(x,t) \quad x < a(t) \quad \text{and} \quad x > 1 \quad . \tag{2}$$

At the leading edge, the values of y given by Equations (1) and (2) will be equal. This occurs at the point which we identify as $x = a(t)$, and from this fact we obtain an important boundary condition:

$$Y(a(t),t) = \left[1 - a(t)\right]\alpha + d + h(t) \quad . \tag{3}$$

For steady motion of the planing surface, let us denote the free-surface elevation by $Y_0(x)$. Then Equations (1), (2), and (3) take the special form:


Y
h(t)
α
α+d
[1 - a(t)]α + d + h(t) = Y(a(t),t)
x
d
a(t)
1
Planing surface, steady motion
Planing surface, unsteady motion
Free surface, steady motion
Free surface, unsteady motion
U

$$y = (1 - x)\alpha + d \quad ; \tag{1'}$$

$$y = Y_0(x) \quad , \quad x < 0 \text{ and } x > 1 \quad ; \tag{2'}$$

$$Y_0(0) = \alpha + d \quad . \tag{3'}$$

Let there be a velocity potential:

$$Ux + \phi(x,y,t) = \underline{\text{Re}}\ Uz + f(z,t) \quad , \tag{4}$$

where $z = x + iy$. The linearized boundary conditions on the potential function are as follows:

$$\phi_y = -\alpha U + \dot{h}(t)$$

$$\equiv v(t) \quad , \text{ on } y = 0 \quad , \quad a(t) < x < 1 \quad ; \tag{5}$$

$$U\phi_x + \phi_t = 0 \quad \text{on} \quad y = 0 \quad , \quad x < a(t) \quad \text{and} \quad x > 1 \ ; \tag{6}$$

$$UY_x + Y_t = \phi_y \quad \text{on} \quad y = 0 \quad , \quad x < a(t) \quad \text{and} \quad x > 1 \ . \tag{7}$$

Equation (5) is the usual kinematic boundary condition on the planing surface. Equations (6) and (7) are, respectively, the dynamic and kinematic conditions on the free surface. Equation (6) can be re-expressed:

$$\phi(x,0,t) = \phi(x_0,0,t-(x-x_0)/U) \quad ,$$

where x_0 is any $x < a(t)$ or $x > 1$. (Different functions must be used in the upstream and downstream regions, of course.) We may suppose that, if we go far enough upstream, there is no disturbance, and so $\phi = 0$

for large enough values of $-x$. If we select x_0 in such a region, we have:

$$\phi(x,0,t) = 0 \quad \text{on} \quad y = 0 \quad , \quad x < a(t) \tag{8}$$

As a corollary, we note that this implies:

$$\phi(x,-y,t) = -\phi(x,y,t) \quad . \tag{8'}$$

It is this result primarily that allows us to reduce our problem to an equivalent-airfoil problem. One must use some care in passing from (6) to (8), because the steady free-surface disturbance is in fact arbitrarily large far away. However, the potential function for steady motion can be chosen so that it is identically zero on $y = 0$ upstream. (This procedure is really justified only in the sense proven by Rispin (1966) and Wu (1967).)

Although Equation (6) applies both upstream and downstream, Equation (8) is valid only for $x < a(t)$, and there does not appear to be any useful way of simplifying (6) for downstream application. However, using (8'), we can extend the definition of the potential function into the whole space, and then Equation (6) becomes just the usual downstream condition for continuity of pressure across a vortex wake. Furthermore, the extended potential must satisfy (5) on $y = +0$ as well as on $y = -0$, and so the body boundary condition is equivalent to that on a flat-plate airfoil of zero thickness. Thus the boundary-value problem for the planing surface is the same as for an oscillating airfoil.

We can integrate (7) in a manner similar to that used on (6):

$$Y(x,t) = Y(x_0, t-(x-x_0)/U) + (1/U) \int_{x_0}^{x} \phi_y(x',0,t-(x-x')/U)\, dx' \quad . \tag{9}$$

The boundary-value problem to be solved then is the following: Given a sinusoidally varying $h(t)$, find the velocity potential for arbitrary amplitude and phase of $a(t)$, the potential satisfying (5), (6), and (8); then use (9) to compute $Y(x,t)$ and finally find $a(t)$ such that (3) is satisfied.

As in the aerodynamic problem, there is a mathematical indeterminacy unless we provide one more condition, namely, the Kutta condition: At the trailing edge, the fluid velocity should be bounded or as weakly singular as possible.

In converting the planing problem into an airfoil problem, one can easily lose sight of the essential physical differences between the two. We shall be solving the problem as if there were a vortex wake, but there is none, of course. There is a free surface instead. We are simply fortunate in that the problem is equivalent mathematically to a well-studied problem. Concepts such as circulation have no place in the planing problem. The fluid region is simply connected and the potential function is single-valued. There is no circulation. Nevertheless, we shall use some of the terms and symbols of aerodynamics in solving the problem.

SOLUTION OF THE BOUNDARY-VALUE PROBLEM

From the equivalent aerodynamic problem, we can expect to find a solution which corresponds to a vorticity distribution over the planing surface and its wake. Therefore we write the complex potential in the form:

$$f(z,t) = \frac{1}{2\pi i}\int_{a(t)}^{\infty}\gamma(\xi,t)\,\log\left[(\xi-z)/\xi\right]d\xi \quad , \tag{10}$$

where

$$-\pi \leq \arg\,(\xi-z) \leq +\pi \quad ;$$

$$-\pi \leq \arg\,\xi \leq +\pi \quad .$$

This form of the solution automatically satisfies (8), the upstream dynamic condition.

The downstream dynamic condition, (6), is satisfied if we require that

$$\gamma(x,t) = \gamma(1+,t-(x-1)/U) \quad \text{for} \quad x > 1 \quad . \tag{11}$$

In the aerodynamic problem, Equation (11) is interpreted to mean that vorticity is convected with the fluid. In the planing problem, it means only that a potential function given as in (10) will represent a fluid motion exhibiting constant pressure on the free surface (in a linearized sense). However, in our extended problem, we have all of the mathematical features of the airfoil problem, including conservation of rotation. Therefore we can proceed strictly mathematically in computing the circulation about the equivalent airfoil; this circulation is time-dependent, and so free vorticity must be generated at the trailing edge. Just as in the airfoil problem, we then have:

$$U\gamma(1+,t) = -\frac{d}{dt}\int_{a(t)}^{1}\gamma(\xi,t)\,d\xi \equiv -\frac{d\Gamma(t)}{dt} \tag{11'}$$

Together with (11), this shows that the function $\gamma(x,t)$ for $x > 1$ is completely determined by the "circulation", $\Gamma(t)$, of the segment from $x = a(t)$ to $x = 1$.

In the interval $a(t) < x < 1$, we find the function $\gamma(x,t)$ in the manner of Karman and Sears (1938). First, let:

$$\gamma(x,t) = \gamma_0(x,t) + \gamma_1(x,t) \quad , \quad a(t) < x < 1 \quad , \tag{12}$$

where

$$\gamma_0(x,t) = 2(-\alpha U+\dot{h})\sqrt{\frac{1-x}{x-a(t)}} = 2v(t)\sqrt{\frac{1-x}{x-a(t)}} \quad . \tag{13}$$

Corresponding to $\gamma_0(x,t)$ we define a potential function:

$$f_0(z,t) = \frac{1}{2\pi i}\int_{a(t)}^{1}\gamma_0(\xi,t)\log\left[(\xi-z)/\xi\right]d\xi \quad . \tag{13a}$$

The real part of this function, say $\phi_0(x,y,t)$, has the property that:

$$\phi_{0_y}(x,0,t) = -\frac{1}{\pi}(-\alpha U+\dot{h}) ⨍_{a(t)}^{1} \sqrt{\frac{1-\xi}{\xi-a}}\,\frac{d\xi}{\xi-x} = -\alpha U+\dot{h} \quad .*$$

Thus $\phi_0(x,0,t)$ by itself satisfies condition (5), and the remainder of the potential must contribute no vertical velocity component on $y = 0$, $a(t) < x < 1$.

The remainder of the complex potential is (cf. (10), (12), (13a).):

$$\frac{1}{2\pi i}\int_{a(t)}^{1} \gamma_1(\xi,t) \log\left[(\xi-z)/\xi\right]d\xi + \frac{1}{2\pi i}\int_{1}^{\infty} \gamma(\xi,t) \log\left[(\xi-z)/\xi\right]d\xi \quad .$$

If we choose $\gamma_1(x,t)$ as follows:

$$\gamma_1(x,t) = \frac{1}{\pi}\sqrt{\frac{1-x}{x-a(t)}} \int_1^{\infty} \sqrt{\frac{\xi-a(t)}{\xi-1}}\;\frac{\gamma(\xi,t)\,d\xi}{\xi-x} \quad , \qquad (14)$$

the required condition is indeed satisfied. This is the classical result of unsteady airfoil theory; see, for example, Karman and Sears (1938). For <u>any</u> wake vorticity distribution, if we choose $\gamma_1(x,t)$ as above in $a(t) < x < 1$, the combination of the wake vorticity and this part of the bound vorticity produces no vertical component of velocity on the airfoil. Mathematically, the same results can be carried over to our planing problem, although the interpretations involving <u>vorticity</u> do not apply.

With $\gamma(x,t)$ given by (11), (11'), (12), (13), and (14), the potential function (10) satisfies three of our boundary conditions, namely, (5), (6), and (8), regardless

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* $⨍$ denotes a Cauchy principal value.

of the nature of the function $a(t)$. In principle, all that remains is to compute the free-surface elevation upstream and match it to the position of the planing surface to determine $a(t)$.

For future use, let us calculate the two parts of the "bound vorticity," that is,

$$\Gamma_j(t) = \int_{a(t)}^{1} \gamma_j(x,t)\, dx \quad . \tag{15}$$

We find:

$$\Gamma_0(t) = \pi(1-a)\,(-\alpha U+\dot{h}) \quad ; \tag{15a}$$

$$\Gamma_1(t) = \int_1^{\infty} d\xi\; \gamma(\xi,t)\left[\sqrt{\frac{\xi-a(t)}{\xi-1}}\; -1\right] \quad . \tag{15b}$$

The "vorticity" just behind the trailing edge is, from (11'),

$$\gamma(1+,t) = -\pi\left[1-a(t)\right]\ddot{h}(t)/U + \pi\dot{a}(t)\left[-\alpha U+\dot{h}(t)\right]/U$$

$$-\frac{1}{U}\int_1^{\infty} d\xi\left(\gamma_t(\xi,t)\left[\sqrt{\frac{\xi-a(t)}{\xi-1}}\; -1\right]\right.$$

$$\left. -\frac{1}{2}\,\gamma(\xi,t)\,\frac{\dot{a}(t)}{\sqrt{(\xi-1)(\xi-a(t))}}\right) \quad . \tag{16}$$

Equation (11) can then be used to specify the function $\gamma(x,t)$ everywhere downstream of the planing surface.

THE SECOND LINEARIZATION

The linearization of the problem required that the steady angle of attack, α, and the motion variables, $h(t)$ and $a(t)$, be small quantities. From Equation (16), it is apparent that we have ended up with formulas which include quadratic functions -- and worse -- of the small quantities. Since we are analyzing the stability of the steady motion of the planing surface, it is consistent with the usual approach to perturbation problems to assume all disturbances to be small enough that only linear combinations of small quantities occur. Let us be specific, however, about the assumptions which are required.

The basic small parameter is α, the angle of attack. As the planing surface heaves an amount h, the location of the forward edge moves a distance which is of the order of magnitude of h/α, that is, $h = O(\alpha a)$. Therefore we require that $h = o(\alpha)$, which implies that $a = o(1)$. The function $\gamma(x,t)$ is also small. Neglecting all terms except those of order $O(h)$, we have the following approximation of Equation (16):

$$\gamma(1+,t) = -\pi\ddot{h}(t)/U - \pi\alpha\dot{a}(t)$$

$$- \frac{1}{U}\int_1^\infty d\xi\ \gamma_t(\xi,t)\left[\sqrt{\frac{\xi}{\xi-1}} - 1\right] \quad . \qquad (16')$$

Now that everything is linear in the time-dependent quantities, we assume that time variation is sinusoidal at frequency ω, and we use the exponential form of the sine function. Let:

$$h(t) = h_0 e^{i\omega t} \quad ; \qquad (17a)$$

$$a(t) = a_0 e^{i(\omega t - \varepsilon)} \quad ; \qquad (17b)$$

$$\gamma(x,t) = g e^{i\omega(t-x/U)} \quad , \quad 1 < x < \infty \quad . \qquad (17c)$$

Note that h_0 and a_0 are real constants, whereas g is generally complex. The new approximation for $\gamma(1+,t)$, in place of (16'), is:

$$\gamma(1+,t) = \Bigg(\pi\omega^2 h_0/U - \pi i\omega\alpha a_0 e^{-i\varepsilon}$$

$$- \frac{gi\omega}{U}\int_1^\infty d\xi e^{-i\omega\xi/U}\left[\sqrt{\frac{\xi}{\xi-1}} - 1\right]\Bigg) e^{i\omega t} \quad . \qquad (18)$$

This equation, with (17c), can be solved for g in terms of h_0 and $a_0 e^{-i\varepsilon}$. Following the usual conventions of unsteady airfoil theory, we write the result:

$$g = -4\pi U e^{i\nu} \left(\frac{i\nu h_0 + \frac{1}{2}\alpha a_0 e^{-i\varepsilon}}{K_0(i\nu) + K_1(i\nu)} \right) , \tag{19}$$

where $\nu = \omega/2U$ and $K_j(z)$ is a modified Bessel function of the second kind. Since the arguments of the Bessel functions are purely imaginary, the functions could be rewritten in terms of Hankel functions (Bessel functions of the third kind) of real argument. With g given as above, the "vorticity" is given everywhere downstream:

$$\gamma(x,t) = \underline{\mathrm{Re}} \left\{ g e^{i\omega(t-x/U)} \right\} , \quad 1 < x < \infty . \tag{20}$$

(We imply the <u>real part</u> of all complex expressions, of course. We shall not generally bother to indicate this explicitly.)

THE CONDITION FOR a(t)

We now return to the upstream kinematic boundary condition, (7) . The right-hand side can be written:

$$\phi_y(x,0,t) = -\frac{1}{2\pi}\int_{a(t)}^{\infty} \gamma(\xi,t)\,\frac{d\xi}{\xi-x}$$

$$= \left[-\alpha U + \dot{h}(t)\right]\left[1 - \sqrt{\frac{1-x}{a(t)-x}}\right]$$

$$- \frac{1}{2\pi}\, g e^{i\omega t}\sqrt{\frac{1-x}{a(t)-x}}\int_1^{\infty}\sqrt{\frac{\xi}{\xi-1}}\;\frac{e^{-i\omega\xi/U}\,d\xi}{\xi-x} \quad . \qquad (21)$$

The integral of the boundary condition, given in (9), can now be computed:

$$Y(x,t) = Y(x_0, t - \frac{x-x_0}{U})$$

$$+ \frac{1}{U}\int_{x_0}^{x} dx'\left(\left[-\alpha U + \dot{h}(\tau)\right]\left[1 - \sqrt{\frac{1-x'}{a(\tau)-x'}}\right]\right.$$

$$\left. - \frac{g}{2\pi}\, e^{i\omega\tau}\sqrt{\frac{1-x'}{a(\tau)-x'}}\int_1^{\infty}\frac{d\xi e^{-i\omega\xi/U}}{\xi-x'}\sqrt{\frac{\xi}{\xi-1}}\right), \qquad (22)$$

where $\tau = t - (x-x')/U$.

We would like to let $x_0 \to -\infty$ in (22), since we may expect that there will be no disturbance there. However, we recall that this solution includes the steady-motion solution, which, far from vanishing, yields an infinite displacement of the free surface far upstream. Nevertheless, we are really interested only in the unsteady part, and one may hope that the latter is not so badly behaved. Let us assume that this is the case. For steady motion, we have the following relationships:

$$\phi_y(x,0,t) = -\alpha U \left[1 - \sqrt{\frac{1-x}{-x}} \right] , \qquad -\infty < x < 0 \quad ;$$

$$Y(x,t) = Y_0(x) = Y_0(x_0) + \frac{1}{U} \int_{x_0}^{x} \phi_y(x',0)dx'$$

$$= Y_0(x_0) - \alpha \int_{x_0}^{x} \left[1 - \sqrt{\frac{1-x'}{-x'}} \right] dx' \quad .$$

We substitute this expression for $Y_0(x)$ into Equation (3'), obtaining:

$$Y_0(x_0) = d + \alpha \left(1 + \int_{x_0}^{0} \left[1 - \sqrt{\frac{1-x'}{-x'}} \right] dx' \right) \quad .$$

In the unsteady-motion problem, we now assume that the displacement of the free surface far upstream is equal to the displacement in the steady-motion problem, i.e., $Y(x_0,t) - Y_0(x_0) \to 0$ as $x \to -\infty$. This is where we explicitly require that the <u>unsteady</u> component of surface displacement vanish upstream. Thus, we substitute this expression for $Y_0(x_0)$ into (22) in place of

$Y(x_0,t)$:

$$Y(x,t) = d + \alpha\left(1 + \int_{x_0}^{0}\left[1-\sqrt{\frac{1-x'}{-x'}}\right]dx'\right)$$

$$+ \frac{1}{U}\int_{x_0}^{x} dx'\left(\left[-\alpha U + \dot{h}(\tau)\right]\left[1-\sqrt{\frac{1-x'}{a(\tau)-x'}}\right]\right.$$

$$\left. - \frac{g}{2\pi} e^{i\omega\tau}\sqrt{\frac{1-x'}{a(\tau)-x'}}\int_{1}^{\infty}\frac{d\xi e^{-i\omega\xi/U}}{\xi-x'}\sqrt{\frac{\xi}{\xi-1}}\right) . \tag{23}$$

Finally, we let x_0 go to $-\infty$, set $x = a(t)$, and then use (3). We obtain the following equation to be solved for $a(t)$:

$$\alpha\left[1-a(t)\right] + d + h(t) = \alpha + d$$

$$+ \alpha \lim_{x_0 \to -\infty}\left(\int_{x_0}^{0}\left[1-\sqrt{\frac{1-x}{-x}}\right]dx - \int_{x_0}^{a(t)}\left[1-\sqrt{\frac{1-x}{a(\tau)-x}}\right]dx\right)$$

$$+ \frac{1}{U}\int_{-\infty}^{0} dx\, \dot{h}(t + \frac{x}{U})\left[1-\sqrt{\frac{1-x}{-x}}\right] \tag{24}$$

$$- \frac{ge^{i\omega t}}{2\pi U}\int_{-\infty}^{0} dx\, e^{i\omega x/U}\sqrt{\frac{1-x}{-x}}\int_{1}^{\infty}\frac{d\xi e^{-i\omega\xi/U}}{\xi-x}\sqrt{\frac{\xi}{\xi-1}} .$$

The limit which appears here does exist if $\omega \neq 0$. Clearly there is a bit of computing to be done to determine $a(t)$, or, what is equivalent, $a_0 e^{-i\varepsilon}$. When the previous expression for g , Equation (19), is substituted into (24), we see that we have simply a linear (complex) equation for $a_0 e^{-i\varepsilon}$, and so the solution, in principle, is trivial.

In the Appendix, it is shown that Equation (24) can be rewritten in terms of standard functions:

$$0 = \alpha a(t) \left(1 - \frac{1}{2} e^{i\nu} K_0(i\nu) + i \int_0^{\nu} d\zeta e^{i\zeta} K_0(i\zeta) \right)$$

$$- i\nu e^{i\nu} h(t) \left[K_1(i\nu) + K_0(i\nu) \right]$$

$$- \frac{1}{2\pi U} g e^{i\omega t} \int_1^{\infty} dx e^{-2i\nu x} \left(\frac{x+1}{x} \right) E\left(\frac{x-1}{x+1} \right) . \qquad (24')$$

where $E(x)$ is the complete elliptic integral of the second kind.

The integral containing the elliptic integral does not exist in a classical sense. However, we can rewrite that term as follows:

$$\int_1^{\infty} dx\ e^{-2i\nu x} \left(\frac{x+1}{x} \right) \quad E\left(\frac{x-1}{x+1} \right)$$

$$= \int_1^{\infty} dx\ e^{-2i\nu x} \left[\frac{x+1}{x} \quad E\left(\frac{x-1}{x+1} \right) - 1 \right] + \frac{\pi}{2} e^{-2i\nu} \left[\delta(\nu) + \frac{1}{\pi i \nu} \right] .$$

In this form, it is obvious that the results are invalid for zero frequency. Here, $\delta(x)$ is the Dirac delta function. Of course, the necessity for an interpretation in terms of generalized functions raises some question about the validity of the results for all frequencies. However, one is certainly not surprised about the difficulty at zero frequency, and the problem formulation seems rather reasonable for finite frequency.

This difficulty can probably be removed in either of two ways: (1) An initial-value problem might be formulated, or (2) the above solution might be interpreted as a near-field solution which ought to be matched to a non-pathological far-field solution.

THE PREDICTION OF INSTABILITY

The lift can be computed by using the momentum approach which is common in aerodynamics, or one may compute the pressure on the planing surface and integrate to find the lift. (In the former case, a factor of one-half must be introduced to eliminate the lift on the upper face of the equivalent airfoil.) The result is the following:

$$L(t) = \pi\rho U^2\alpha + \left[A_1(\nu) + iA_2(\nu)\right] e^{i\omega t} \tag{25}$$

where

$$A_1(\nu) + iA_2(\nu) = -\pi\rho U^2 \left(i\nu h_0 \left[i\nu + \frac{2K_1(i\nu)}{K_0(i\nu)+K_1(i\nu)} \right] + \alpha a_0 e^{-i\varepsilon} \left[i\nu + \frac{K_1(i\nu)}{K_0(i\nu)+K_1(i\nu)} \right] \right) . \tag{25'}$$

The lift is, of course, just the real part of this expression, and the heave motion is given by:

$$h(t) = h_0 \cos \omega t \quad ; \quad \dot{h}(t) = -\omega h_0 \sin \omega t \quad . \qquad (26)$$

Thus we can rewrite the lift expression:

$$L = \pi \rho U^2 \alpha + \frac{A_1(\nu)}{h_0} h(t) + \frac{A_2(\nu)}{\omega h_0} \dot{h}(t) \quad . \qquad (27)$$

If $A_2(\nu) < 0$, the oscillation is stable, but, for $A_2(\nu) > 0$, the hydronamic lift will provide negative damping and instability will occur. It is a numerical problem to determine under what conditions $A_2(\nu)$ becomes positive.

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APPENDIX

The mathematical details are presented here for the step from Equation (24) to Equation (24'). There are three integral terms in (24), which we discuss in order.

The first term is:

$$I_{x_0} = \alpha \lim_{x_0 \to -\infty} \left(\int_{x_0}^{0} \left[1 - \sqrt{\frac{1-x}{-x}} \right] dx - \int_{x_0}^{a(t)} \left[1 - \sqrt{\frac{1-x}{a(\tau)-x}} \right] dx \right) ,$$

where

$$\tau = t - \left[a(t) - x \right] / U \quad ,$$

$$a(t) = a_0 \, e^{i(\omega t - \varepsilon)} \quad .$$

In the following paragraphs, it is understood that the symbol a always means $a(t)$ unless we specifically write $a(\tau)$.

First we approximate the square-root involving $a(\tau)$; the following sequence of steps is rather obvious:

$$\left[a(\tau) - x\right]^{-\frac{1}{2}} = \left[a(t)\exp\left[-i\omega(a-x)/U\right] - x\right]^{-\frac{1}{2}}$$

$$= \left((a-x) - a + a\exp\left[-i\omega(a-x)/U\right]\right)^{-\frac{1}{2}}$$

$$= \left[a(t)-x\right]^{-\frac{1}{2}}\left(1 - a(t)\left[\frac{1 - \exp\left[-i\omega(a-x)/U\right]}{a-x}\right]\right)^{-\frac{1}{2}} .$$

The second term in the second factor on the right-hand side can always be made less than 1 in magnitude if we make a small enough. To show this, choose some number m such that $|2a(t)/m| < 1$. The quantity under consideration is an entire function of the complex variable z, if we simply replace x by z:

$$\frac{1 - \exp\left[-i\omega(a-z)/U\right]}{a-z} .$$

Therefore it has a power series which converges in the whole z-plane. For $|z-a| < m$, there exists a number M such that

$$\frac{1 - \exp\left[-i\omega(a-z)/U\right]}{a-z} < M .$$

We can ensure that $|Ma(t)| < 1$ just by making $a(t)$ small enough. For $|a-x| > m$, the following estimate is valid:

$$\left| \frac{1 - \exp[-i\omega(a-x)/U]}{a-x} \right| < \frac{2}{m} \quad .$$

Thus for either $|x-a| < m$ or $|x-a| > m$, we have that:

$$|a(t)| \left| \frac{1 - \exp[-i\omega(a-x)/U]}{a-x} \right| < 1 \quad .$$

We can now expand the second factor in the following series:

$$\left(1-a(t) \left[\frac{1 - \exp[-i\omega(a-x)/U]}{a-x}\right]\right)^{-1/2}$$

$$= 1 + \frac{a(t)}{2} \left[\frac{1 - \exp[-i\omega(a-x)/U]}{a-x}\right] + O(a^2) \quad .$$

The integral which includes this factor can be written:

$$\int_{x_0}^{a(t)} \sqrt{\frac{1-x}{a(\tau)-x}}\, dx = \int_{x_0}^{a(t)} dx \sqrt{\frac{1-x}{a(t)-x}} \left(1 + \frac{a(t)}{2}\left[\frac{1-\exp[-i\omega(a-x)/U]}{a-x}\right]\right.$$

$$\left. + O(a^2) \right) \quad ,$$

We change the variable of integration:

$$x' = (x-a)/(1-a/x_0) \quad ,$$

and reorder terms in the expression for I_{x_0} , obtaining:

$$I_{x_0} = -\frac{\alpha a(t)}{2} \lim_{x_0 \to -\infty} \left(\log 4|x_0| - \int_{x_0}^{0} dx \sqrt{\frac{1-x}{-x}} \left[\frac{1-e^{+i\omega x/U}}{-x} \right] \right) .$$

The integral can be transformed into standard form by another change of variable: $\xi = 2x+1$. Let $\nu = \omega/2U$. Then:

$$I_{x_0} = -\frac{\alpha a(t)}{2} \lim_{x_0 \to -\infty} \left(e^{i\nu} \int_{1}^{2|x_0|+1} \frac{d\xi e^{-i\nu\xi}}{\sqrt{\xi^2-1}} - 2\int_{1}^{2|x_0|+1} \frac{d\xi}{\sqrt{\xi^2-1}} \frac{1-e^{-i\nu(\xi-1)}}{\xi-1} \right) .$$

We can now let $x_0 \to -\infty$. The first term is just $e^{i\nu}K_0(i\nu)$. We note that the derivative of the second term with respect to ν is:

$$2 \frac{d}{d\nu} \int_{1}^{\infty} \frac{d\xi}{\sqrt{\xi^2-1}} \frac{1-e^{-i\nu(\xi-1)}}{\xi-1} = 2i \int_{1}^{\infty} \frac{d\xi}{\sqrt{\xi^2-1}} e^{-i\nu(\xi-1)}$$

$$= 2ie^{i\nu}K_0(i\nu) .$$

Thus,

$$2\int_{1}^{\infty} \frac{d\xi}{\sqrt{\xi^2-1}} \frac{1-e^{-i\nu(\xi-1)}}{\xi-1} = 2i \int_{0}^{\nu} d\zeta e^{i\zeta} K_0(i\zeta) ,$$

since the left-hand side is zero for $\nu = 0$. Combining these results, we have:

$$I_{x_0} = -\frac{\alpha a(t)}{2}\left[e^{i\nu}K_0(i\nu) - 2i\int_0^\nu d\zeta e^{i\zeta}K_0(i\zeta)\right] .$$

The second integral term in Equation (24) can be reduced rather simply by the following steps:

$$\frac{1}{U}\int_{-\infty}^{0} dx\ \dot{h}(t+\tfrac{x}{U})\ \left[1-\sqrt{\frac{1-x}{-x}}\right]$$

$$= \frac{1}{U}\ \dot{h}(t)\ \int_{-\infty}^{0} dx\ e^{i\omega x/U}\left[1-\sqrt{\frac{1-x}{-x}}\right]$$

$$= \frac{1}{U}\ \dot{h}(t)\left(\frac{1}{2i\nu} - \frac{1}{2}\ e^{i\nu}\left[K_0(i\nu)+K_1(i\nu)\right]\right) .$$

We may note that

$$\frac{1}{2i\nu U}\ \dot{h}(t) = \frac{1}{i\omega}\ \dot{h}(t) = h(t) ,$$

and so the first term above cancels another one of the terms in (24).

The third integral in (24) can be put into a symmetrical form by a simple change of variable:

$$\int_{-\infty}^{0} dx\ e^{i\omega x/U}\sqrt{\frac{1-x}{-x}}\int_1^\infty \frac{d\xi e^{-i\omega\xi/U}}{\xi-x}\sqrt{\frac{\xi}{\xi-1}}$$

$$= \frac{1}{2}\int_1^\infty dx\ e^{-i\nu x}\sqrt{\frac{x+1}{x-1}}\int_1^\infty \frac{d\xi}{\xi+x}\ e^{-i\nu\xi}\sqrt{\frac{\xi+1}{\xi-1}} .$$

We now substitute:

$$y = (x+\xi)/2 \quad , \quad \eta = (x-\xi)/2 \quad .$$

The differential changes: $dx\, d\xi = 2\, dy\, d\eta$. The y -integration extends from 1 to ∞ , and the η -integration from $-(y-1)$ to $(y-1)$; however, the integrand is even with respect to η , and so we can cut the range of η in half and multiply the integral by 2 . For the above integral, we now have:

$$\int_1^\infty \frac{dy}{y}\, e^{-2i\nu y} \int_0^{y-1} d\eta \sqrt{\frac{(y+1)^2-\eta^2}{(y-1)^2-\eta^2}}$$

$$= \int_1^\infty dy\, e^{-2i\nu y} \left(\frac{y+1}{y}\right) \int_0^1 dt \sqrt{\frac{1-k^2t^2}{1-t^2}} \quad ,$$

where $k = (y-1)/(y+1)$. The inner integral is now just the elliptic integral (complete) of the second kind. This completes the derivation of (24').

It was noted following Equation (24') that the integral containing the elliptic integral in its integrand does not really exist in terms of conventional functions. The generalized-function interpretation is based on the concepts developed in, for example, Lighthill (1958).

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A mathematical analysis is developed for the hydrodynamic problem of a two-dimensional planing surface which is heaving sinusoidally. From the assumptions of small angle of attack and small amplitudes of motion, it is possible to formulate a linear problem. Gravity is neglected. A condition is developed for predicting instability of the steady forward motion.

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